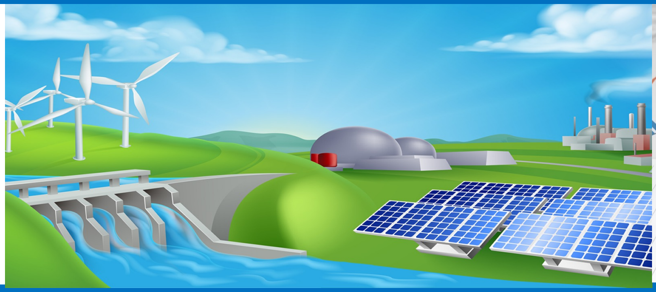



# INTRO TO LINEAR PROGRAMMING (LP)

Prof. Luana Medeiros Marangon Lima, Ph.D.

# Intro to LP

# To express a problem in terms of mathematical programming you need:

1. An objective (what to optimize)
2. Alternative actions
3. Limited resources
4. Variables need to be related
5. It needs to be possible to express 1, 2, and 3 in mathematical language through equations or inequalities

# Example:

A production plant manufactures two types of water heaters (type 1 & 2). Find the production schedule (number of units type 1 and 2 to be manufactured) that maximizes profits.

- profits for selling type 1 = $800, per unit
- profits for selling type 2 = $600, per unit
- Both types need to be processed in two different machines in any order
  - Type 1 requires:
    - 4 hours of processing in machine A
    - 2 hours in machine B
  - Type 2 requires:
    - 2 hours in machine A
    - 4 hours in machine B
  - There is a limited number of processing hours available at each machine
    - 60 hours for machine A
    - 48 hours for machine B

Can this be formulated as a mathematical program?

# What do we want to optimize?

- We want to maximize profits!!

# What are the limited resources?

- The time of machines A and B

# What do we need to decide?

- How many units of water heater type I and type II

# Solution

- Graphically
- Algebraically
- Computer software

# Graphical method

1. **Formulate problem**
2. Construct a graph and plot constraint lines
3. Determine feasible region
4. Find optimal solution
   - Calculate value of objective function for all corners and choose optimal

   Or

   - Plot two objective function lines to determine direction of improvement. Choose corner and calculate value of objective function

# Step 1: Formulating the problem

1. Identify decision variables and objective function. Name your variables

   *$z$: profits*

   *$x_1$: number of units of water heater type 1 to produce*

   *$x_2$ : number of units of water heater type 2 to produce*

2. Write equation for objective function

$$\max_{x_1,x_2} z = \$800\, x_1 + \$600 x_2$$

# Very important

- We will formulate this optimization problem as one of **linear programming (LP)**

- This means that
  - Our decision variables will be represent by linear relationships
  - And we assume that the decision variables can be any positive real number

# Formulating the problem

1. Identify decision variables and objective function. Name your variables
2. Write equation for objective function
3. Identify constraints
   - Constraint 1: Cannot use more than the available hours at machine A
   - Constraint 2: Cannot use more than the available hours at machine B
4. Write equations to represent constraints

# Equation stating we cannot use more than the available hours for Machine A

$$4x_1 + 2x_2 \leq 60$$

Hours used with production of WH Type I

Hours used with production of WH Type II

# Formulating the problem

1. Identify decision variables and objective function. Name your variables
2. Write equation for objective function
3. Identify constraints
   - Constraint 1: Cannot use more than the available hours at machine A
   - Constraint 2: Cannot use more than the available hours at machine B
4. Write equations to represent constraints
   - C1: $4x_1 + 2x_2 \leq 60$
   - C2: $2x_1 + 4x_2 \leq 48$

# Formulating the problem

1. Identify decision variables and objective function. Name your variables
2. Write equation for objective function
3. Identify constraints
4. Write equations to represent constraints
5. Write the non-negativity constraint
   - C3: $x_1, x_2 \geq 0$

# Final LP Formulation

- Maximizing profits in the water heaters' production plant

$$\max_{x_1, x_2} \quad z = \$800\, x_1 + \$600 x_2$$

$$s.t. \quad 4x_1 + 2x_2 \leq 60$$

$$2x_1 + 4x_2 \leq 48$$

$$x_1, x_2 \geq 0$$

- *where*

*$z$: profits*

*$x_1$: number of units of water heater type 1 to produce*

*$x_2$ : number of units of water heater type 2 to produce*

# Understanding the Notation

$$\max_{x_1,x_2} \quad z = \$800\, x_1 + \$600 x_2$$

Profit coefficients

$$\begin{aligned} s.t. \quad & 4x_1 + 2x_2 \le 60 \\ & 2x_1 + 4x_2 \le 48 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Technical coefficients

Right hand side

General Formulation

$$\max_{x_1,x_2} \quad z = \boldsymbol{c_1} x_1 + \boldsymbol{c_2} x_2$$

$$\begin{aligned} s.t. \quad & a_{11}x_1 + a_{12}x_2 \le b_1 \\ & a_{21}x_1 + a_{22}x_2 \le b_2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

- *where*

  $z$: *profits*

  $x_1$: *number of units of water heater type 1 to produce*

  $x_2$ : *number of units of water heater type 2 to produce*

# Graphical method

1. Formulate problem
2. **Construct a graph and plot constraint lines**
3. Determine feasible region
4. Find optimal solution

- Calculate value of objective function for all corners and choose optimal

Or

- Plot two objective function lines to determine direction of improvement. Choose corner and calculate value of objective function

# Step 2: Construct a graph and plot constraint lines

- Construct a graph:
  - Let the x axis represent your first decision variable and the y axis the other decision variable
- Plot constraint lines
  - Transform constraint equations into equalities

$$4x_1 + 2x_2 = 60$$
$$2x_1 + 4x_2 = 48$$
$$x_1, x_2 = 0$$

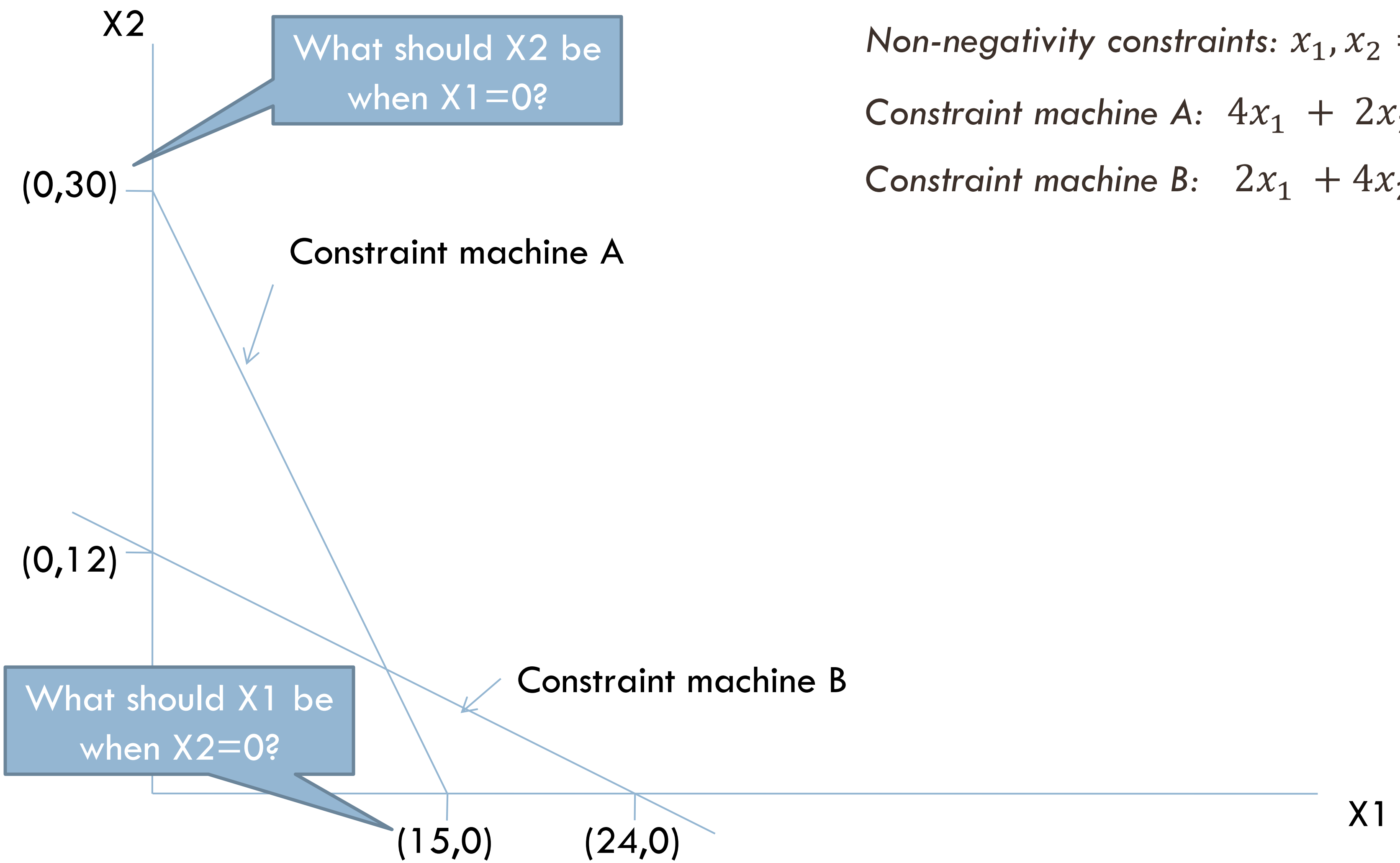
  - Find the intersection points with the x,y axes. For each constraint line ask:
    - What is the value of $x_1$ when $x_2$ is 0
    - What is the value of $x_2$ when $x_1$ is 0

# Graphically: Plot constraints

*Non-negativity constraints:* $x_1, x_2 = 0$

*Constraint machine A:* $4x_1 + 2x_2 = 60$

*Constraint machine B:* $2x_1 + 4x_2 = 48$



# Graphical method

1. Formulate problem
2. Construct a graph and plot constraint lines
3. **Determine feasible region**
4. Find optimal solution

- Calculate value of objective function for all corners and choose optimal

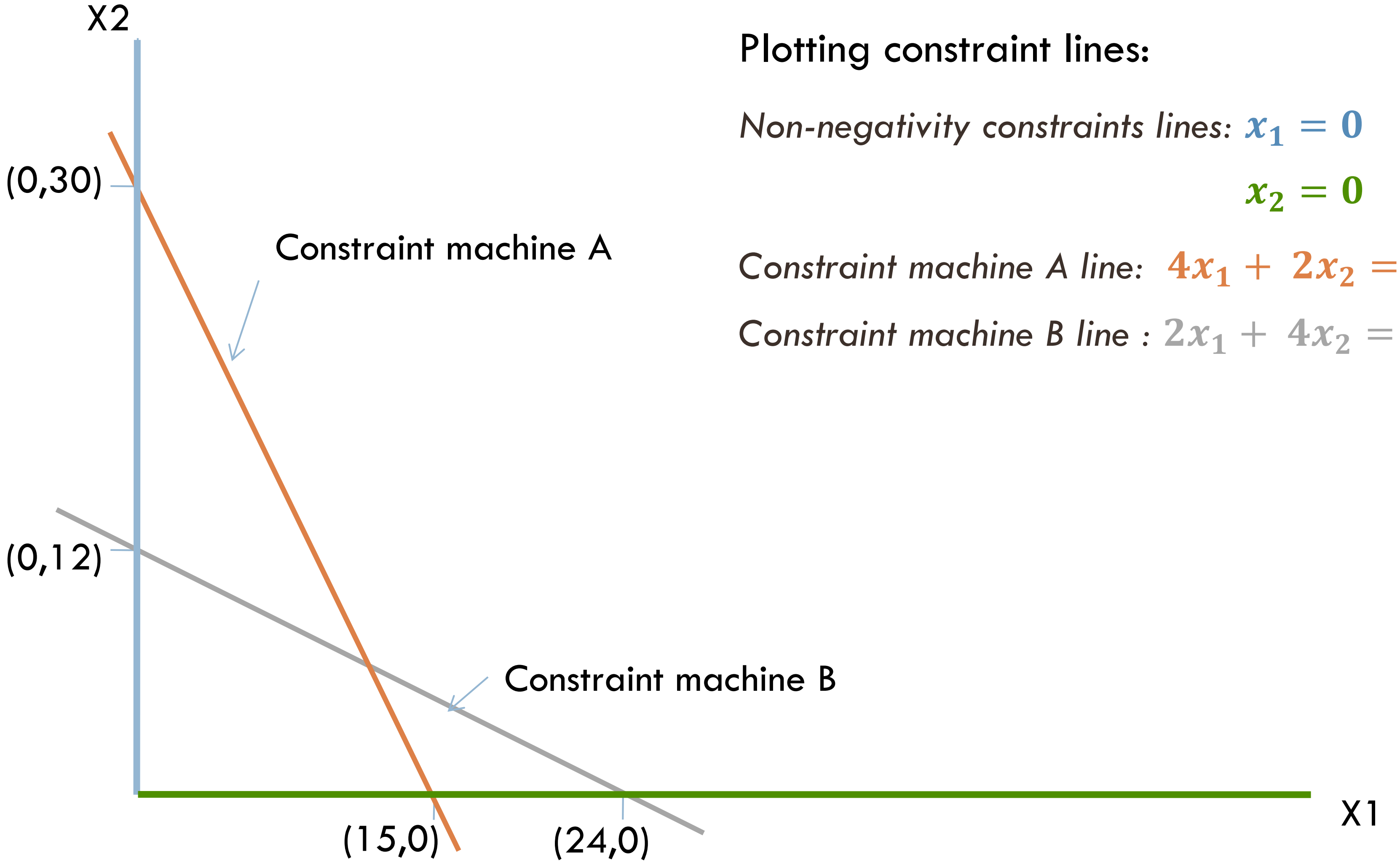
Or

- Plot two objective function lines to determine direction of improvement. Choose corner and calculate value of objective function

# Step 3: Determine Feasible Region

- Determine the feasible (valid) side of each constraint line
- Feasible region is the area of the graph valid for all constraints
- Since the solution needs to meet all constraints → the solution is in the feasible region

# Determine Feasible Region




Plotting constraint lines:

*Non-negativity constraints lines:* $x_1 = 0$

$x_2 = 0$

*Constraint machine A line:* $4x_1 + 2x_2 = 60$

*Constraint machine B line :* $2x_1 + 4x_2 = 48$

# Determine Feasible Region

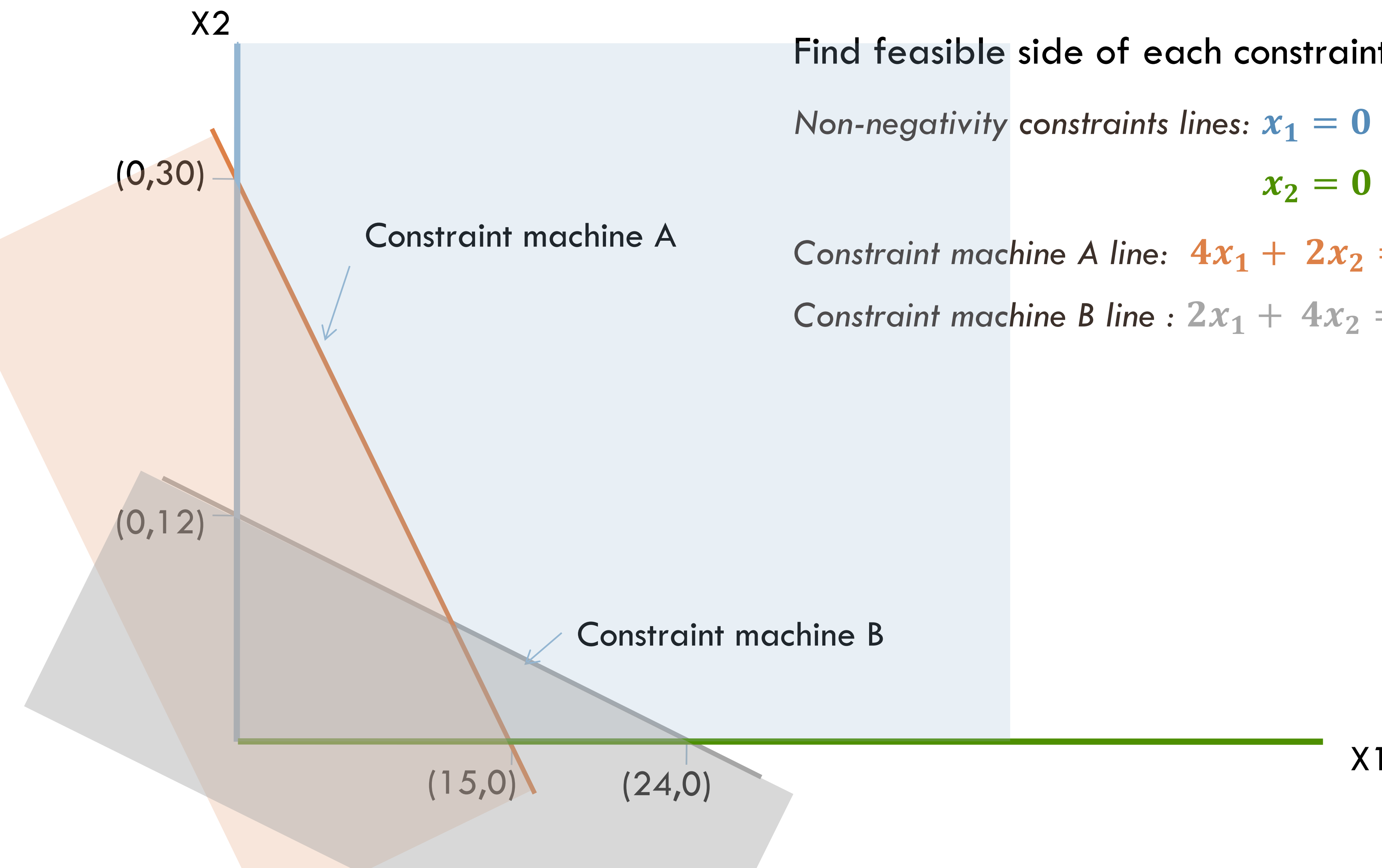
Find feasible side of each constraint:

*Non-negativity constraints lines:* $x_1 = 0$

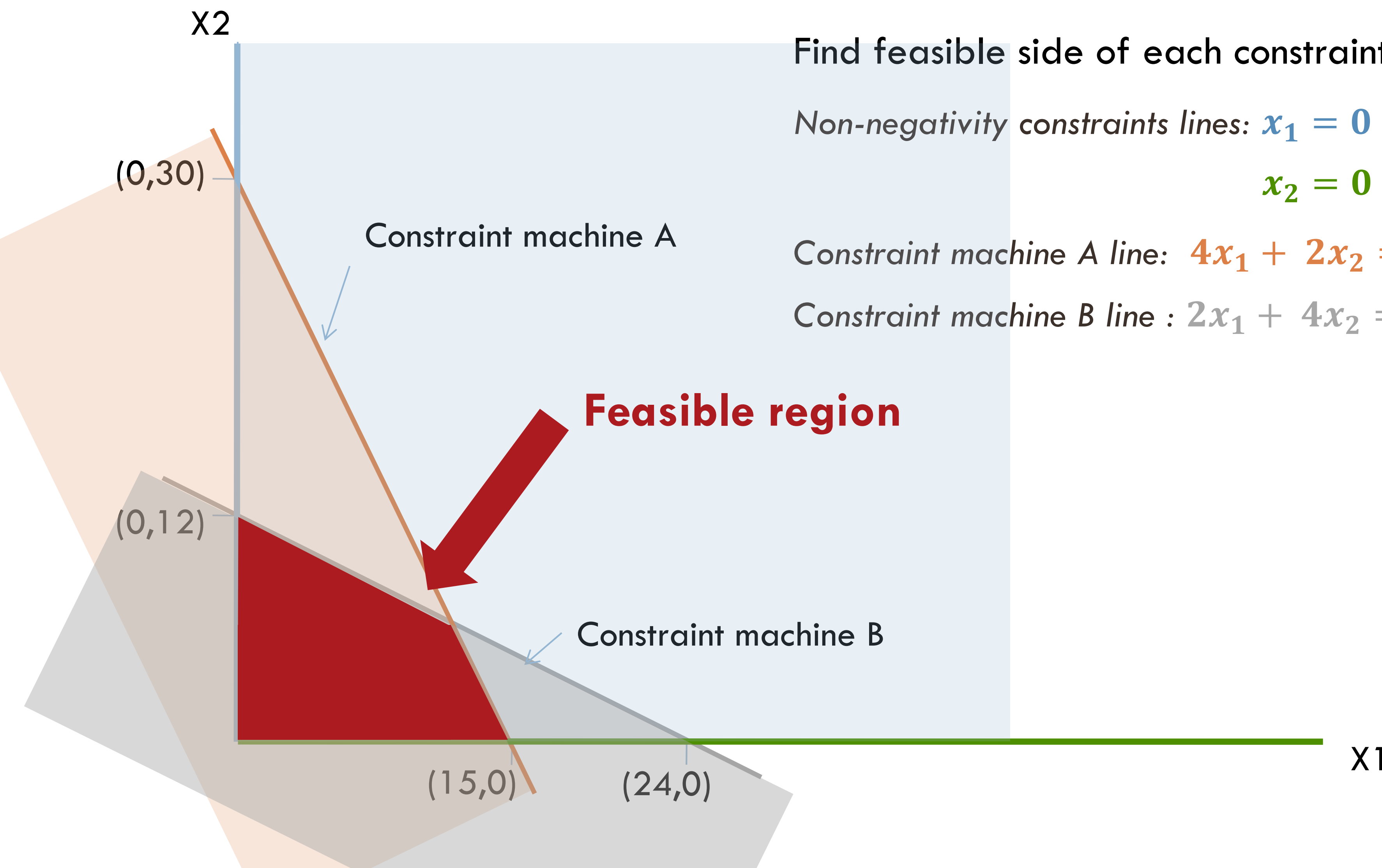
$x_2 = 0$

*Constraint machine A line:* $4x_1 + 2x_2 = 60$

*Constraint machine B line :* $2x_1 + 4x_2 = 48$



# Determine Feasible Region




Find feasible side of each constraint:

*Non-negativity constraints lines:* $x_1 = 0$

$x_2 = 0$

*Constraint machine A line:* $4x_1 + 2x_2 = 60$

*Constraint machine B line :* $2x_1 + 4x_2 = 48$

# Graphical method

1. Formulate problem
2. Construct a graph and plot constraint lines
3. Determine feasible region
4. **Find optimal solution**

- Calculate value of objective function for all corners and choose optimal

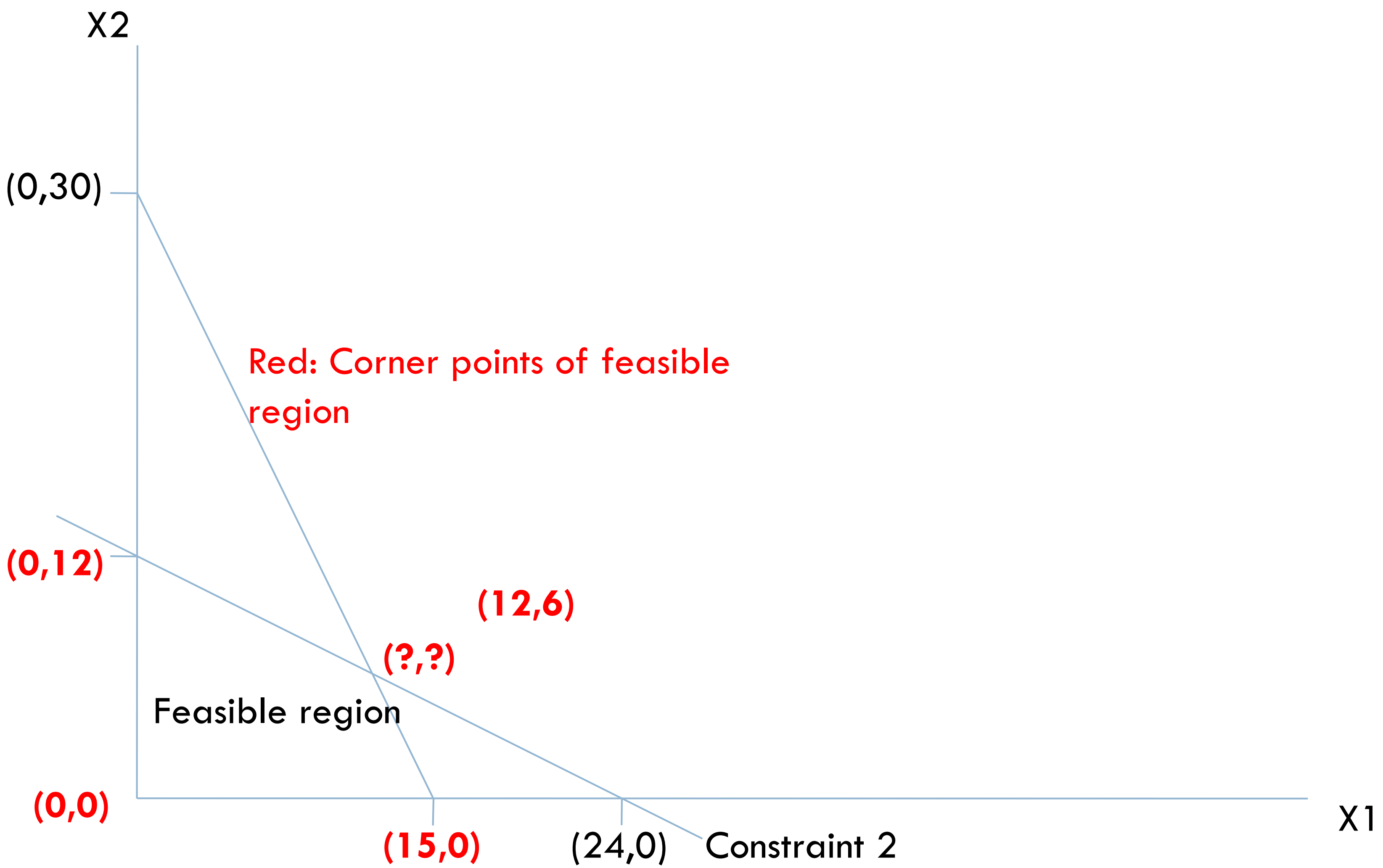
Or

- Plot two objective function lines to determine direction of improvement. Choose corner and calculate value of objective function

# Step 4: Find optimal solution

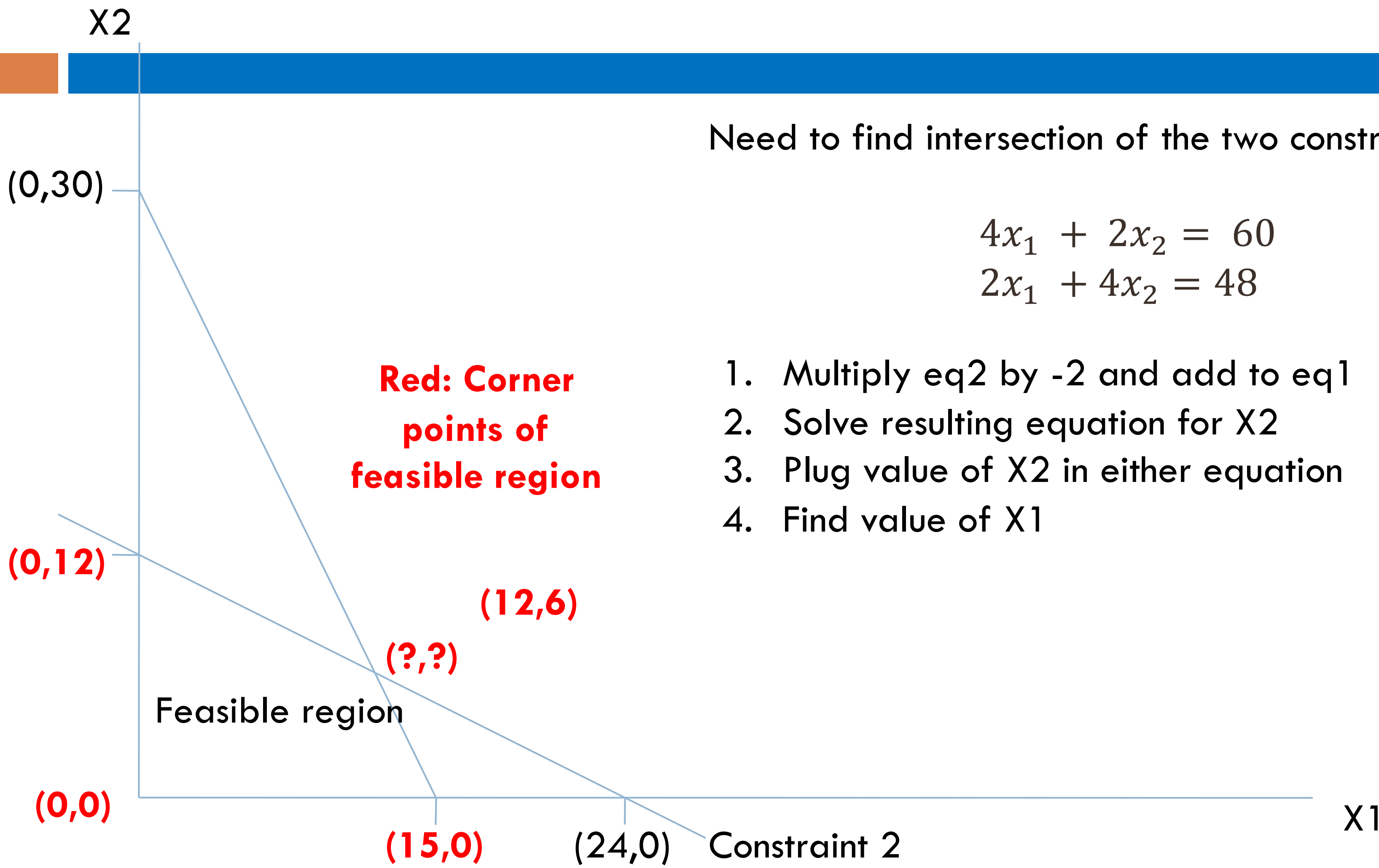
- Locate extreme points in the feasible region
- Calculate objective function for all corners

# Find Corner Points



# How de we find the intersection point?

# Find Corner Points




Need to find intersection of the two constraint lines:

$$4x_1 + 2x_2 = 60$$
$$2x_1 + 4x_2 = 48$$

1. Multiply eq2 by -2 and add to eq1
2. Solve resulting equation for X2
3. Plug value of X2 in either equation
4. Find value of X1

# Calculate objective function for all red corners

| **Feasible solutions: (X1,X2)** | **Hours used** | | **Profits (Z)** |
|---|---|---|---|
| | **Machine A =4X1+2X2** | **Machine B =2X1+4X2** | **=$800X1+$600X2** |
| (0,0) | 0 | 0 | 0 |
| (0,12) | 2*12=24 | 4*12=48 | $600*12=$7,200 |
| (15,0) | 4*15=60 | 2*1=30 | $800*15=$12,000 |
| (12,6) | 4*12+2*6=60 | 2*12+4*6=48 | $800*12+$600*6=$13,200 |

# Graphical method

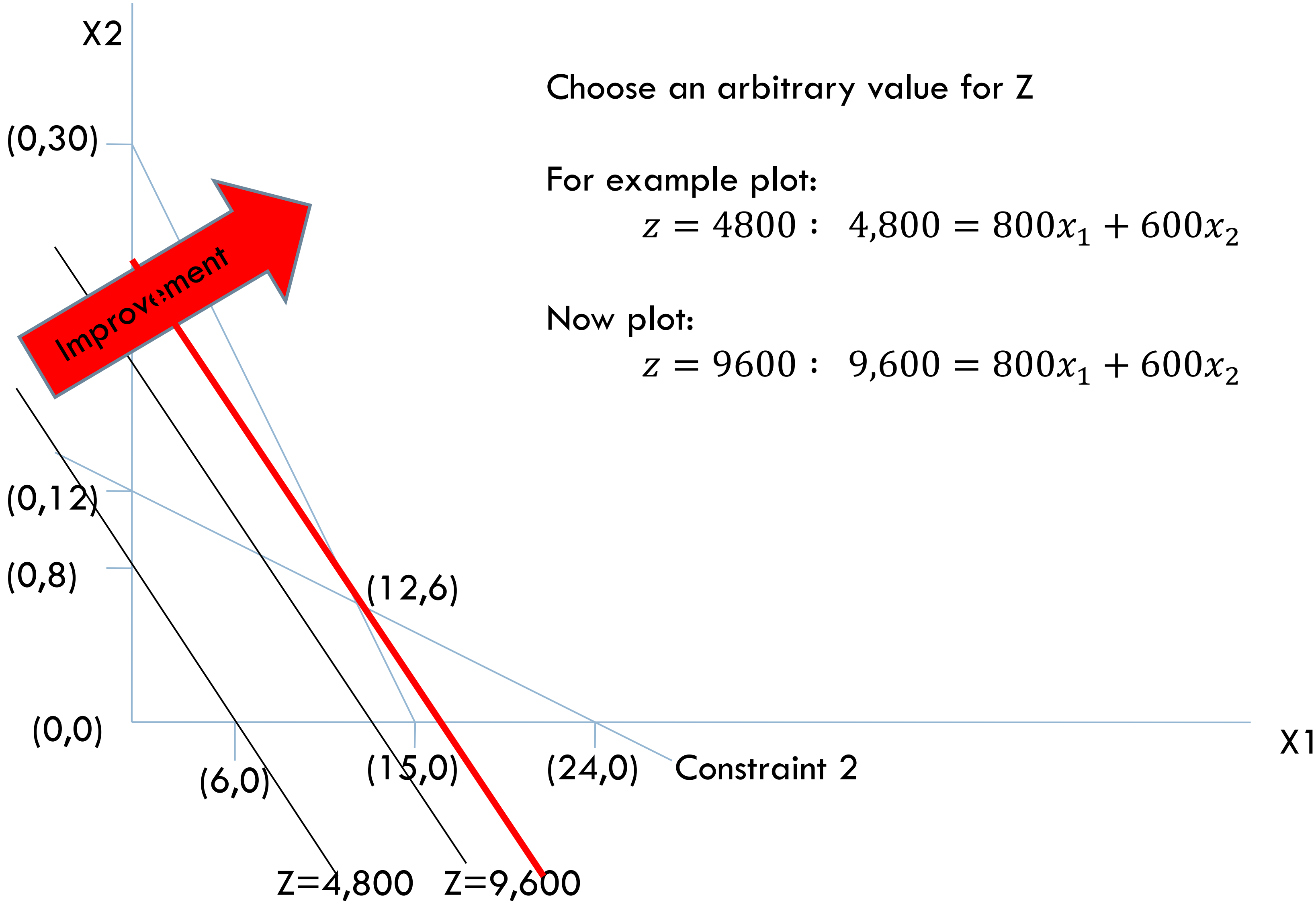
1. Formulate problem
2. Construct a graph and plot constraint lines
3. Determine feasible region
4. **Find optimal solution**

- Calculate value of objective function for all corners and choose optimal

Or

- Plot two objective function isolines to determine direction of improvement. Choose corner and calculate value of objective function

# Alternative to evaluating Z at all corner points: Plot objective function and see direction of improvement




Choose an arbitrary value for Z

For example plot:

$$z = 4800: \quad 4{,}800 = 800x_1 + 600x_2$$

Now plot:

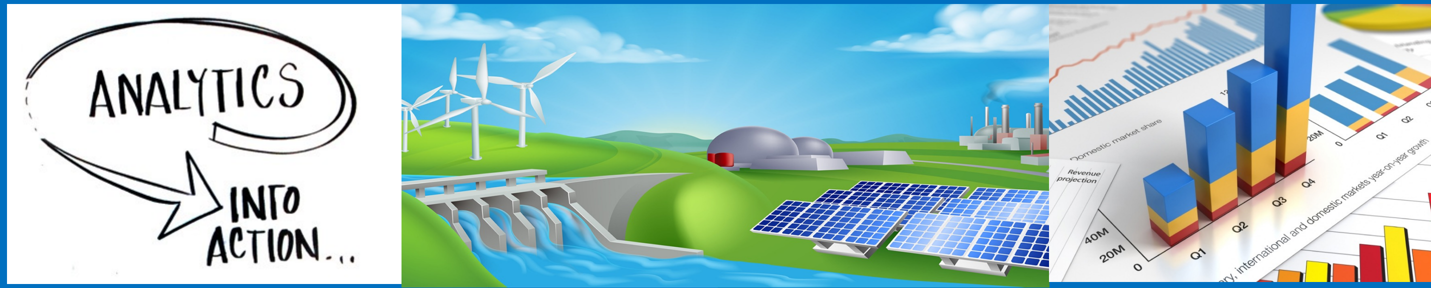
$$z = 9600: \quad 9{,}600 = 800x_1 + 600x_2$$

# LP concepts

- Feasible region
- Feasible solution
- Objective function
- Decision variables
- Constraint (binding and non binding)

# LP assumptions

- Proportionality: Contribution to costs (or profits) of a variable is linear. No returns to scale /economies of scale
- Additivity: Total cost is sum of individual cost
- Divisibility: Decision variables can be divided into fractional levels (e.g. can make 5.2 water heaters)
- Deterministic: The cost coefficients, technical coefficients and RHS are all known deterministically




# THANK YOU !

luana.marangon.lima@duke.edu